The Sylvester–Hadamard game matrix does not have the largest boosting gap: a counterexample to Conjecture 2 of Nie, Warmuth, Vishwanathan and Zhang

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Abstract

Nie, Warmuth, Vishwanathan and Zhang (COLT 2013) view boosting as a zero-sum game and build game matrices from Hadamard matrices. Let $\hat{H}$ be the $n \times n$ Hadamard matrix with its first row removed. Their Conjecture 2 states that $\hat{H}$ gives the largest gaps among all $(n-1) \times n$ matrices with entries ± 1 and game value 0: for every such matrix U and every $1 \le t \le n$, the best t-column sub-matrix of $\hat{H}$ has a game value at most that of the best t-column sub-matrix of U. We show that the conjecture is false when H is the Sylvester Hadamard matrix of order $n = 16$. For $t = 6$, the matrix U obtained from the Hadamard matrix Had.16.3 of Sloane's library by deleting its all-ones row satisfies $\max_{U_6} \mathrm{val}_D(U_6) = -3/7$, while $\max_{\hat{H}_6} \mathrm{val}_D(\hat{H}_6) \ge -1/3$. The upper bound for U comes from 8008 exact dual certificates, one for each set of 6 columns. Lean 4 with Mathlib checks the full statement from the definitions, with kernel evaluation of the certificates and only the standard axioms. Two exact checkers in Python also accept the certificates. The counterexample does not refute the weaker reading in which $\hat{H}$ can come from any Hadamard matrix of order n. This work has not yet been reviewed by experts. The computations, the proofs and the text were produced by AI models (from Anthropic and OpenAI), working under the author's direction.

1 Introduction

Boosting can be viewed as a zero-sum game between a row player and a column player [1]. The rows of a game matrix are the examples, and the columns are the hypotheses. After t iterations, a boosting algorithm has seen t columns, and the value of the sub-matrix U_t of these columns is the largest minimum margin that a convex combination of them can reach. The source [1] asks how slowly this value can approach the value of the full matrix. It defines the gap at step t as $\mathrm{val}(U) - \max_{U_t} \mathrm{val}(U_t)$, where the maximum is over all sub-matrices of t columns. A large gap for every t gives a lower bound for every boosting algorithm.

The source builds its game matrices from Hadamard matrices. For $n = 2^k$ it takes the $n \times n$ Hadamard matrix H, deletes the first row to get $\hat{H}$, and uses the game matrix $\begin{bmatrix} \hat{H} \\ -\hat{H} \end{bmatrix}$. It proves a lower bound of $\sqrt{1/(2t)}$ on the gap for $1 \le t \le n/2$ (its Theorem), and states two conjectures. Conjecture 1 asserts a stronger lower bound of order $\sqrt{(\log n)/t}$ for $\log n \le t \le cn$. Conjecture 2 asserts that $\hat{H}$ gives the largest gaps among all ± 1 matrices of value 0. The source reports that its authors verified Conjecture 2 "by tedious combinatorial arguments for $n = 2, 4, 8$ and $t \le n$ as well as for $n = 2^k$ and $n - 2 \le t \le n$".

Prior work. Grønlund, Larsen and Mathiasen [2] refuted Conjecture 1. Their paper says "Our main contribution is a refutal of this conjecture", and their algorithm SparsiBoost refutes the conjectured lower bound. Their paper does not mention Conjecture 2. It is the only work in OpenAlex that cites [1]. We found no claim of a settlement of Conjecture 2 in our searches (Section 6).

Result. Conjecture 2 is false for $n = 16$ and $t = 6$ when H is the Sylvester Hadamard matrix (Theorem 1). The counterexample U is itself $\hat{H}'$ for a different Hadamard matrix H' of order 16. So the conjecture fails because the gap of the Hadamard construction depends on the choice of the Hadamard matrix. The case $(n, t) = (16, 6)$ is outside the range that the source verified.

2 Definitions and the conjecture

We copy the definitions of the source [1]. For $t \geq 1$ let $S^t = \{w \in \mathbb{R}^t : w_j \geq 0,\ \sum_j w_j = 1\}$ be the probability simplex. For a matrix G with t columns, equation (1) of the source defines

$$\mathrm{val}(G) = \min_{d} \max_{w \in S^t} d^\top G w = \max_{w \in S^t} \min_{r} [Gw]_r.$$

For a matrix U the source writes $\mathrm{val}_D(U) = \mathrm{val}\left(\left[\begin{smallmatrix} U \\ -U \end{smallmatrix}\right]\right)$ and notes that

$$\mathrm{val}_D(U) = -\min_{w \in S^t} \|Uw\|_\infty \leq 0.$$

Let $n = 2^k$, let H be the $n \times n$ Hadamard matrix, and let $\hat{H}$ be H with its first row removed. The conjecture is quoted word for word:

Conjecture 2 of Nie, Warmuth, Vishwanathan and Zhang ([1, p. 3])**.** *For any $(n-1) \times n$ dimensional ± 1 valued matrix U satisfying $\mathrm{val}_D(U) = 0$, the following inequality holds for $1 \leq t \leq n$: $\max_{\hat{H}_t} \mathrm{val}_D(\hat{H}_t) \leq \max_{U_t} \mathrm{val}_D(U_t)$, where $\hat{H}_t$ is any t column sub-matrix of $\hat{H}$ and U_t is any t column sub-matrix of U.*

The Hadamard matrix. The source does not give a construction of "the" Hadamard matrix. For $n = 2^k$ we use the Sylvester matrix $H_{rc} = (-1)^{\langle r,c \rangle}$, $0 \leq r, c < n$, where $\langle r, c \rangle$ is the number of common 1 bits of the binary forms of r and c. Its first row is all $+1$, so $\hat{H}$ has row sums 0 and $\mathrm{val}_D(\hat{H}) = 0$, as the source's proof of its Theorem requires. This reading is the one that we refute. Remark 1 discusses the other readings.

A "t column sub-matrix" is the sub-matrix on t distinct columns. Repeated columns would not change the maximum for U: a repeated column adds nothing to the convex hull of the columns.

3 The counterexample

Let U be the 15×16 matrix in Figure 1. It is the Hadamard matrix Had.16.3 of Sloane's library [3] with its row 0 deleted. Row 0 of Had.16.3 is all $+1$, so every row of U has sum 0 and $[\mathbf{1}^\top; U]$ is a Hadamard matrix. Written with $+$ for $+1$ and $-$ for -1, the rows of U (rows 1 to 15 of Had.16.3, read left to right and then down) are

```
+-+-+-+-+-+-+-+-  ++--++--++--++--  +--++--++--++--+
++++----++++----  +-+--+-++-+--+-+  ++----+++----++
+--+-++-+--+-++-  +++++++++-------  +++-+------+-+++
++-+---+--+-+++-  ++---++---+++--+  +-++-+---+--+-++
+-+---++-+-+++--  +--++-+--++--+-+  +---++-+-++--+-
```

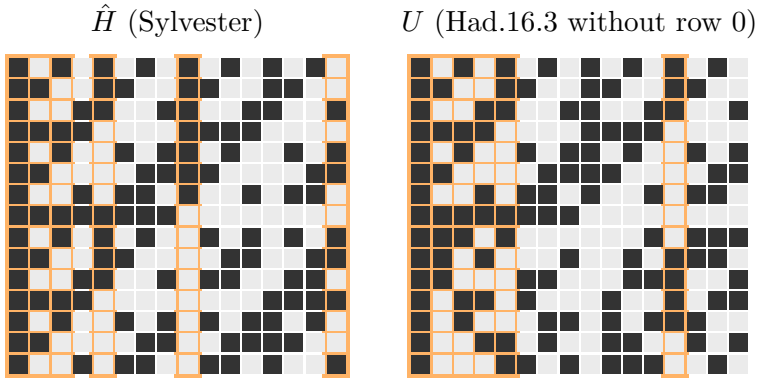


Figure 1: Left: $\hat{H}$ from the Sylvester matrix of order 16. Right: the counterexample U. Dark cells are $+1$, light cells are -1. The shaded columns are $S_0 = \{0, 1, 2, 4, 8, 15\}$ (left), where $\|\hat{H}_{S_0} w\|_\infty = 1/3$ for uniform w, and $\{0, 1, 2, 3, 4, 12\}$ (right), where U reaches its optimum $3/7$.

Theorem 1. *Let $n = 16$, let H be the Sylvester Hadamard matrix and let U be as above. Then U is a 15×16 matrix with entries ± 1 and $\mathrm{val}_D(U) = 0$, and*

$$\max_{\hat{H}_6} \mathrm{val}_D(\hat{H}_6) \;\geq\; -\tfrac{1}{3} \;>\; -\tfrac{3}{7} \;=\; \max_{U_6} \mathrm{val}_D(U_6).$$

So Conjecture 2 is false for $n = 16$ and $t = 6$.

The proof has three parts.

$\mathrm{val}_D(U) = 0$. Every row of U sums to 0. For the uniform vector $w = \mathbf{1}/16$ we get $Uw = 0$, so $\mathrm{val}_D(U) \geq 0$. Also $\mathrm{val}_D(U) \leq 0$ always.

The Sylvester side. Take the columns $S_0 = \{0, 1, 2, 4, 8, 15\}$. As vectors of $\mathbb{F}_2^4$ they are 0, the four unit vectors and 1111. For a row $r \neq 0$ with $|r|$ ones, the sum of the row of $\hat{H}$ over S_0 is

$$1 + \textstyle\sum_{i=1}^{4} (-1)^{r_i} + (-1)^{|r|} = 1 + (4 - 2|r|) + (-1)^{|r|} \in \{2, -2\}.$$

With $w = \mathbf{1}/6$ this gives $\|\hat{H}_{S_0} w\|_\infty = 1/3$, so $\max_{\hat{H}_6} \mathrm{val}_D(\hat{H}_6) \geq \mathrm{val}_D(\hat{H}_{S_0}) \geq -1/3$. A floating-point solution of all 8008 linear programs for $\hat{H}$ gives $\min_S \min_w \|\hat{H}_S w\|_\infty = 0.333333$, so equality holds in floating point. The proof uses only the inequality.

The upper bound for U. The bound comes from weak duality.

Lemma 2. *Let A be an $m \times t$ matrix, $c > 0$ and $y \in \mathbb{R}^m$ with $y \neq 0$. If $(A^\top y)_j \geq c\,\|y\|_1$ for every column j, then $\|Aw\|_\infty \geq c$ for every $w \in S^t$, so $\mathrm{val}_D(A) \leq -c$.*

Proof. For $w \in S^t$, $\|y\|_1 \|Aw\|_\infty \geq y^\top Aw = \sum_j w_j (A^\top y)_j \geq c\|y\|_1 \sum_j w_j = c\|y\|_1$. □

For each of the $\binom{16}{6} = 8008$ sets S of 6 columns we give a vector y_S with $(U_S^\top y_S)_j \geq \frac{3}{7}\|y_S\|_1$ for every $j \in S$. Lemma 2 then gives $\mathrm{val}_D(U_S) \leq -3/7$ for every S. We found the vectors y_S as dual solutions of the linear programs $\min_{w \in S^6} \|U_S w\|_\infty$ (HiGHS through SciPy) and rounded them to rational numbers. After scaling, every y_S is an integer vector with entries in $\{-2, \ldots, 2\}$; 840

of them have one nonzero entry, and none has more than six. The condition is then an integer inequality, $7\,(U_S^\top y_S)_j \geq 3\|y_S\|_1$, which a computer checks exactly.

Equality holds: for $S = \{0, 1, 2, 3, 4, 12\}$ and $w = (1, 1, 1, 1, 1, 2)/7$ we get $\|U_S w\|_\infty = 3/7$ exactly, so $\max_{U_6} \mathrm{val}_D(U_6) = -3/7$. This completes the proof.

Remark 1 (Which Hadamard matrix). Up to equivalence there are five Hadamard matrices of order 16, listed in Sloane's library as Had.16.0 to Had.16.4 [3]. For each of them and each choice of the deleted row (after column sign changes that make that row all $+1$), we solved all 8008 linear programs at $t = 6$ in floating point. Had.16.0, Had.16.1 and Had.16.2 give the value 0.333333 for all 16 rows; Had.16.3 and Had.16.4 give 0.428571 for all 16 rows. The Sylvester matrix is a Hadamard matrix of order 16, so Theorem 1 alone also refutes the reading "for every Hadamard matrix H of order n". It does not refute the reading "for some Hadamard matrix H of order n": we did not test whether a matrix U beats $\hat{H}$ from Had.16.3 or Had.16.4 at some t. These floating-point values are information only; Theorem 1 does not use them.

4 Formal proof in Lean

The file `lean/Solution.lean` proves the theorem

```
theorem HadamardBoosting.conjecture2_false : ¬ Conjecture2 4
```

in Lean 4 [4] with Mathlib [5] (Lean and Mathlib v4.36.0). The file `lean/Challenge.lean` repeats the definitions and the statement. The definitions follow the source directly:

- `sylvester k` is the matrix $(-1)^{\langle r,c\rangle}$ of order 2^k, and `hhat k` is `sylvester k` without row 0;
- `simplex` is S^t, and `val G` is $\sup_{w \in S^t} \min_r [Gw]_r$, the second form of equation (1);
- `valD U` is `val` of the stacked matrix $[U; -U]$ (`Matrix.fromRows`);
- `maxValD U t` is the supremum of `valD` over the sub-matrices on sets of t distinct columns (a `Finset` of size t);
- `Conjecture2 k` is the statement of Conjecture 2 for $n = 2^k$, with U an integer matrix whose entries are 1 or -1.

The source writes max; the Lean file writes the supremum `sSup`. The two are equal here because the simplex is compact and the sets of columns are finite, but the proof does not need this fact. It proves the bounds of Theorem 1 directly for the supremum.

The proof has a general part and a computed part. The general part proves Lemma 2 and the bounds on `valD` and `maxValD` for real weights. The computed part checks the certificates with `decide +kernel`, that is, by evaluation in the Lean kernel, with no extra axiom. To keep each kernel evaluation small, a script (`checks/make_lean.py`) groups the certificates. For each pair $a < b$ of the two smallest columns there is a short list of certificates that covers every set $\{a < b < c < d < e < f\}$; there are 66 such pairs and 2222 list entries in total. Each entry encodes the scaled vector y and the set of columns j with $7(U^\top y)_j \geq 3\|y\|_1$. For each pair, one theorem checks this inequality for every entry, and a second theorem checks the cover. The command `#print axioms` gives only `propext`, `Classical.choice` and `Quot.sound`. The tool `lake comparator` confirms that the proof file proves exactly the theorem of `Challenge.lean`. On one core of a Steam Deck (AMD Zen 2) the proof file compiles in about 20 minutes, with up to about 6 GB of memory; most of the time is the kernel evaluation of the certificates.

5 How the result was checked

1. `checks/check.py`: exact rational arithmetic (Python `fractions`), standard library only. It checks the size and entries of U, the row sums, the 8008 certificates, and the Sylvester witness.
2. `checks/vcheck.py`: a second checker, written from the source definitions without reading the first. It repeats the exact checks, solves the 8008 linear programs again with HiGHS, makes its own dual vectors, rounds them, and checks them exactly (0 failures). It also finds the equality case $S = \{0, 1, 2, 3, 4, 12\}$.
3. `checks/check_extra.py`: U equals Had.16.3 [3] without row 0, the equality case, and the hand formula for the Sylvester row sums.
4. `lean/Solution.lean`: the formal proof (Section 4).
5. `checks/make_lean.py` regenerates `lean/Solution.lean` from the certificate file, so the Lean data and the Python data are the same.

The script `checks/ci.sh` runs items 1, 2, 3 and 5 and compares the outputs with the numbers of this paper.

6 Limits

- The result refutes Conjecture 2 for the Sylvester matrix, and for the reading "every Hadamard matrix". It does not refute the reading in which the Hadamard matrix of order n can be chosen (Remark 1).
- We give one pair $(n, t) = (16, 6)$. We do not know if Conjecture 2 fails for other pairs. The source's own verification ($n \leq 8$, and $n - 2 \leq t \leq n$) does not conflict with our result.
- The result says nothing about the asymptotic lower bound of Conjecture 1, which [2] refuted.
- Novelty. We searched two collections of AI-produced results, GitHub, Zenodo, OpenAlex (one citing work, [2]), arXiv and the web, in two separate passes on 27 November 2026. The Semantic Scholar API returned HTTP 429 in each of our attempts, and we did not search Google Scholar. The PhD thesis of the first author of the source was not readable to us. A settlement of Conjecture 2 that we did not find is therefore possible.

References

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[3] N. J. A. Sloane. A Library of Hadamard Matrices. `http://neilsloane.com/hadamard/` (file `had.16.3.txt`).

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