The configuration LP of restricted assignment need not have a half-integral point at the LP makespan

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Abstract

Encz, Mastrolilli and Vercesi conjectured that for every instance P of the restricted assignment problem, the configuration LP $\mathrm{CLP}(P, \mathrm{LP}(P))$ at the smallest feasible integer makespan $\mathrm{LP}(P)$ contains a half-integral point (their Conjecture 10). If it were true, the integrality gap of the configuration LP would reduce to the graph balancing case. We show that the conjecture is false. Our counterexample has 12 machines and 20 jobs with processing times 1 and 3, and $\mathrm{LP}(P) = 3$. The total processing time is $36 = 12 \cdot 3$, so every configuration in the support of a feasible point has load exactly 3. Each such configuration contains 0 or 3 jobs of a fixed set of 4 jobs. In a half-integral point these 4 jobs are covered 8 times in total, with weights $1/2$, and 8 is not a multiple of 3. Lean 4 with Mathlib checks the disproof from the definitions, with only the standard axioms. We also give a second counterexample with 5 machines and 14 jobs, found by a random search and checked by an exhaustive search. The counterexample does not change the known bounds on the integrality gap.

This work has not yet been reviewed by experts. The computations, the proofs and the text were produced by AI models (from Anthropic and OpenAI), working under the author's direction.

Keywords. Restricted assignment; configuration LP; half-integrality; integrality gap; scheduling; Lean.

1 Introduction

In the restricted assignment problem, n jobs must be assigned to m machines. Job j has a processing time $p_j \in \mathbb{N}_+$ and a set $\mathcal{M}_j$ of machines on which it can run. The goal is to minimize the makespan, the largest total processing time on a machine. In scheduling notation the problem is $P|\mathcal{M}_j|C_{\max}$.

The configuration LP is a strong relaxation of this problem. Its integrality gap is open. Encz, Mastrolilli and Vercesi [1] state that work on the restricted assignment framework places the gap $\mathrm{IG}(P|\mathcal{M}_j|C_{\max})$ between $3/2$ and $11/6$; they cite Svensson [4], Jansen and Rohwedder [2], and Jansen, Land and Maack (reference [14] of [1]). For the graph balancing case, where every job is allowed on at most two machines, they state that the gap lies between $3/2$ and 1.749, the upper bound by Jansen and Rohwedder [3].

In [1] the authors develop integrality gap preserving reductions and apply them to several problems, among them the configuration LP of restricted assignment. In their concluding section they write that the lack of an understanding of the vertices of this LP prevented a deeper analysis, and they state the following conjecture, which they "experimentally asserted for several small-scale instances" [1, p. 23:10].

Conjecture 1 ([1, Conjecture 10])**.** *For every instance P of the restricted assignment problem, the polyhedron* $\mathrm{CLP}(P, \mathrm{LP}(P))$ *contains a half-integral feasible solution; that is, a vector x such that* $x_{i,C} \in \{0, \frac{1}{2}, 1\}$ *for all* $i \in [m]$ *and* $C \in \mathcal{C}_i(P, \mathrm{LP}(P))$.

The authors also report that a stronger version, half-integrality of every vertex, was disproved by their experiments, and that the version above had resisted refutation [1, p. 23:10]. They explain the motivation: in a half-integral point each job lies in at most two configurations, so all other machines can be removed from $\mathcal{M}_j$ without a change of the gap. If the conjecture held, the integrality gap of the configuration LP of restricted assignment would therefore equal that of graph balancing, and the bound 1.749 of [3] would follow [1, p. 23:10].

Theorem 1. *Conjecture 10 of [1] is false. The instance P^* of Section 3 has 12 machines, 20 jobs and* $\mathrm{LP}(P^*) = 3$, *and* $\mathrm{CLP}(P^*, 3)$ *has no half-integral point.*

The proof is short (Section 3.1). It is a parity-type argument modulo 3. A Lean formalization checks it from the definitions (Section 5.1). Section 4 gives a second, smaller counterexample that was found by a computer search. The source paper is from September 2026. We know of no other work that discusses Conjecture 10.

2 Definitions

We use the definitions of [1, Section 6]. An instance is a matrix $P = (p_{j,i})$ for jobs $j \in [n]$ and machines $i \in [m]$. In restricted assignment, $p_{j,i} = p_j \in \mathbb{N}_+$ for $i \in \mathcal{M}_j$ and $p_{j,i} = \infty$ otherwise. For an integer T, the configurations of machine i are

$$\mathcal{C}_i(P, T) = \Big\{ C \subseteq [n] : \sum_{j \in C} p_{j,i} \le T \Big\}. \tag{1}$$

A configuration is a set of jobs, so it contains each job at most once. Its load is at most T, not necessarily equal to T. A configuration of machine i that contains a job j with $i \notin \mathcal{M}_j$ has infinite load, so every configuration of i consists of jobs allowed on i. We write $p(C) = \sum_{j \in C} p_j$. The polyhedron $\mathrm{CLP}(P, T)$ is the set of vectors $x = (x_{i,C})_{i \in [m],\, C \in \mathcal{C}_i(P,T)}$ with

$$\sum_{C \in \mathcal{C}_i(P,T)} x_{i,C} = 1, \qquad i \in [m], \tag{2}$$

$$\sum_{i \in [m]} \sum_{C \in \mathcal{C}_i(P,T) \colon j \in C} x_{i,C} = 1, \qquad j \in [n], \tag{3}$$

$$x_{i,C} \ge 0, \qquad i \in [m],\ C \in \mathcal{C}_i(P, T). \tag{4}$$

$\mathrm{LP}(P)$ is the smallest integer T for which $\mathrm{CLP}(P, T)$ is feasible [1, p. 23:8]. A point is half-integral if all its entries lie in $\{0, \frac{1}{2}, 1\}$.

The following identity is standard. For a point $x \in \mathrm{CLP}(P, T)$ and weights $w \colon [n] \to \mathbb{R}$, write $w(C) = \sum_{j \in C} w_j$. Then

$$\sum_{i \in [m]} \sum_{C \in \mathcal{C}_i(P,T)} x_{i,C}\, w(C) = \sum_{j \in [n]} w_j. \tag{5}$$

To prove it, exchange the sums and apply (3). With $w = p$, (2), (4) and $p(C) \le T$ give the volume bound

$$\sum_{j \in [n]} p_j = \sum_i \sum_C x_{i,C}\, p(C) \le \sum_i \sum_C x_{i,C}\, T = mT. \tag{6}$$

3 The counterexample

The machines are the pairs (c, t) with a *copy* $c \in \{0, 1, 2\}$ and $t \in \{0, 1, 2, 3\}$, so $m = 12$. There are two kinds of jobs, so $n = 20$.

- *Point jobs* $P(c, k)$ for $c \in \{0, 1, 2\}$, $k \in \{0, 1, 2, 3\}$: size 1, allowed on the three machines (c, t) with $t \neq k$.
- *Filler jobs* $F(f)$ for $f \in \{0, \ldots, 7\}$: size 3. Machine (c, t) allows $F(2t)$ and $F(2t+1)$. So $F(f)$ is allowed on the three machines $(0, \lfloor f/2 \rfloor)$, $(1, \lfloor f/2 \rfloor)$, $(2, \lfloor f/2 \rfloor)$.

Each machine allows exactly 5 jobs: 3 point jobs of its own copy and 2 fillers. Within one copy, the 4 machines correspond to the 4 subsets of size 3 of the 4 point jobs (Figure 1). The total processing time is $12 \cdot 1 + 8 \cdot 3 = 36$.

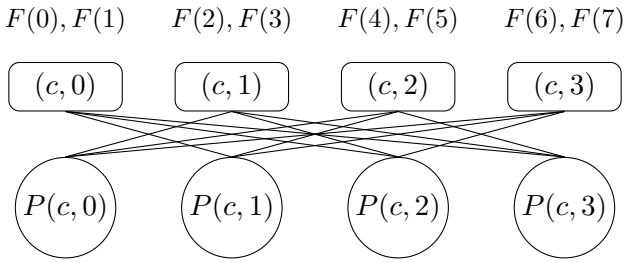


Figure 1: One copy c of the instance. A line joins a machine and a point job that it allows. Above each machine are its two filler jobs; these are shared by the machines $(0, t)$, $(1, t)$, $(2, t)$.

3.1 The proof

Lemma 2. $\mathrm{LP}(P^*) = 3$.

Proof. If $T \leq 2$, the volume bound (6) would give $36 \leq 12T \leq 24$, so $\mathrm{CLP}(P^*, T)$ is empty. For $T = 3$, let x have the value $\frac{1}{3}$ on every configuration of load exactly 3, and 0 on all other configurations. A configuration of load exactly 3 on machine (c, t) is either a single filler, or the set of its three point jobs, because a filler and a point job together have load 4. So each machine has exactly 3 such configurations, and (2) holds. Each filler lies in such a configuration on each of its 3 machines. Each point job $P(c, k)$ lies in the three-point configuration of each of its 3 machines. So (3) holds, and x is a point of $\mathrm{CLP}(P^*, 3)$. □

Within one copy, this point puts weight $\frac{1}{3}$ on each of the 4 subsets of size 3 of a 4-set. These are the only full configurations that contain point jobs, and the fillers fill the remaining capacity.

Lemma 3. *If* $x \in \mathrm{CLP}(P^*, 3)$ *and* $x_{i,C} > 0$, *then* $p(C) = 3$.

Proof. The total processing time is $36 = 12 \cdot 3 = mT$. So (6) holds with equality, that is, $\sum_i \sum_C x_{i,C} (3 - p(C)) = 0$. Each term is nonnegative, so each term is zero. □

Lemma 3 is where configurations of load less than 3 are excluded. Such configurations exist (for example a single point job), and they can contain 1 or 2 point jobs of copy 0. The tight volume shows that they get weight 0 in every feasible point.

Proof of Theorem 1. Let $w_j = 1$ if j is a point job $P(0,k)$ of copy 0, and $w_j = 0$ otherwise; so $\sum_j w_j = 4$. Every configuration C with $p(C) = 3$ satisfies $w(C) \in \{0,3\}$: a single filler has $w(C) = 0$, and the three point jobs of machine (c,t) have $w(C) = 3$ if $c = 0$ and 0 otherwise. Suppose that $x \in \mathrm{CLP}(P^*, 3)$ is half-integral, and let $y = 2x$. Then y is a vector of nonnegative integers. By (5) with the weights w,

$$\sum_i \sum_C y_{i,C}\, w(C) = 2 \sum_j w_j = 8.$$

By Lemma 3, every term with $y_{i,C} \neq 0$ has $p(C) = 3$, so $w(C) \in \{0,3\}$. So the left side is a multiple of 3. But 8 is not a multiple of 3. So $\mathrm{CLP}(P^*, 3) = \mathrm{CLP}(P^*, \mathrm{LP}(P^*))$ has no half-integral point. □

Remark 4. The same argument excludes every point of $\mathrm{CLP}(P^*, 3)$ whose entries are rationals with denominators prime to 3: if d is such a common denominator, then $\sum y\, w = 4d$ for $y = dx$, and $3 \nmid 4d$. In particular $\mathrm{CLP}(P^*, 3)$ has no integral point. An explicit assignment with makespan 4 exists: machines $(0,t)$ and $(1,t)$ take $F(2t)$ and $F(2t+1)$ respectively, and each machine (c,t) takes $P(c, t+1 \bmod 4)$. So $\mathrm{IP}(P^*) = 4$, and the integrality gap of P^* is $4/3$. This is less than $3/2$, so P^* does not improve the known lower bound on the integrality gap.

The construction uses three copies for a reason of volume. In the point of Lemma 2, the triples carry weight $\frac{1}{3}$ on each of the 4 machines, and the other $\frac{2}{3}$ of each machine must be filled by fillers of size 3. Each filler needs total weight 1, so the number of fillers is $\frac{2}{3} \cdot 4c$ for c copies. This is an integer for $c = 3$, which gives the 8 fillers.

4 A second counterexample

A random walk among small instances with an integrality gap found a second counterexample. It has 5 machines (indexed from 0), 14 jobs and $\mathrm{LP}(P) = 14$. For the jobs $j = 0, \ldots, 13$, the processing times are

$$(p_0, \ldots, p_{13}) = (1, 5, 1, 5, 1, 6, 1, 10, 8, 3, 4, 9, 6, 8),$$

and the sets $\mathcal{M}_0, \ldots, \mathcal{M}_{13}$ are

$$\{1,4\},\ \{1,2,4\},\ \{2,3\},\ \{1,4\},\ \{2,3,4\},\ \{1,2,3,4\},\ \{0,1,3\},$$
$$\{3,4\},\ \{0,1,2,3\},\ \{2\},\ \{2,4\},\ \{0,1,2\},\ \{0,1,2,3,4\},\ \{1\}.$$

The total processing time is $68 > 5 \cdot 13 = 65$, so $\mathrm{CLP}(P, 13)$ is empty by (6). The file `instance_small.json` in the folder `checks/` gives a point of $\mathrm{CLP}(P, 14)$ with 12 nonzero entries, all equal to $\frac{1}{3}$ or $\frac{2}{3}$. The machines have 9, 74, 105, 49 and 70 configurations at $T = 14$. An exhaustive search shows that $\mathrm{CLP}(P, 14)$ has no half-integral point. We have no short proof for this instance, and it is not checked in Lean. The volume here is not tight ($68 < 70$), so the argument of Section 3.1 does not apply directly.

5 How the result was checked

5.1 Lean

The file `lean/Solution.lean` states the definitions of Section 2 in Lean 4 with Mathlib (both at v4.36.0) and proves the theorem `HalfIntegralCLP.not_conjecture10`, whose statement is ¬ `Conjecture10`. The definitions are as follows.

- An instance has n jobs `Fin n` and m machines `Fin m`, a size function p: `Fin n` $\to \mathbb{N}$ with $p_j > 0$, and sets $\mathcal{M}_j$.
- The configurations $\mathcal{C}_i(P, T)$, for an integer T, are the subsets of the jobs allowed on i whose load is at most T. This is the reading of (1) with $p_{j,i} = \infty$ for $i \notin \mathcal{M}_j$.
- A point is a real function $x_{i,C}$. The constraints (2)–(4) refer only to the entries with $C \in \mathcal{C}_i(P, T)$.
- `Conjecture10` says: for all n, m, all instances P and all integers T, if T is the least element of $\{T \in \mathbb{Z} : \mathrm{CLP}(P, T) \neq \emptyset\}$, then $\mathrm{CLP}(P, T)$ has a point with every entry $x_{i,C}$, $C \in \mathcal{C}_i(P, T)$, in $\{0, \frac{1}{2}, 1\}$.

The proof follows Section 3.1: identity (5) and the volume bound are proved for all instances. The finite facts about P^* (the total size 36, the 3 full configurations of each machine, the coverage of each job, and $w(C) \in \{0, 3\}$ for each full configuration) are proved by evaluation in the Lean kernel (`decide +kernel`). The command `#print axioms` reports only `propext`, `Classical.choice` and `Quot.sound`. The file compiles in about 18 s on one core after Mathlib is built. The file `lean/Challenge.lean` repeats the definitions and the statement with `sorry`, and `lake comparator` confirms that the proof proves exactly this statement.

5.2 Python

The folder `checks/` has exact checks that use only the Python standard library (integers and `fractions.Fraction`).

- `check_certificate.py` checks the point of Lemma 2, the volume facts, and $w(C) \equiv 0 \pmod 3$ for all 36 configurations of load 3 (3 on each of the 12 machines).
- `vcheck.py` is a second checker, written separately from the first. It enumerates $\mathcal{C}_i(P, T)$ from (1) (10 configurations per machine for P^* at $T = 3$), decides the feasibility of $\mathrm{CLP}(P, T)$ and $\mathrm{CLP}(P, T-1)$ with an exact rational simplex method, and decides whether a half-integral point exists with an exhaustive search. The search does not use the argument modulo 3. For P^* it visits 2227 nodes and finds no point; as a control, it finds a half-integral point at $T = 4$. For the second instance it visits 155269 nodes and finds no point.
- `control.py` compares the pruned search of `vcheck.py` with a search without pruning on 400 random small instances for each of the seeds 1 and 2. All answers agree (282 and 281 of the instances have a half-integral point).
- `check_small.py` checks the second counterexample again, with the point check and the search of `check_counterexample.py`, which share no code with `vcheck.py`.
- `check_ip.py` checks the assignment with makespan 4.

6 Limits

- The result disproves Conjecture 10 as stated. It does not change the known bounds on the integrality gap of the configuration LP for restricted assignment. The instance P^* has gap 4/3.

- The result closes the route of [1, p. 23:10] from half-integral points to graph balancing. It does not show that the gap of restricted assignment differs from that of graph balancing.
- We did not study weaker forms of the conjecture or other routes from restricted assignment to graph balancing.
- Conjecture 10 appeared in September 2026. We found no other work that discusses it. Its importance comes from the reduction that its authors describe.
- The second counterexample relies on an exhaustive computer search. Only the first counterexample is checked in Lean.

References

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