No spectrally minimal graph of order $2k$ with matching number k lies in Hou and Tan's family $\mathcal{T}'_{2k,k}$

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Abstract

Let $\mathcal{G}_{n,k}$ be the family of connected graphs of order n with matching number k. Hou and Tan (arXiv:2607.11061v2) proved a sharp lower bound for the spectral radius on $\mathcal{G}_{n,k}$. For general $n \geq 2k$ they defined a family $\mathcal{T}'_{n,k}$ of trees (Construction 9) and asked: for arbitrary n and k, must every spectrally minimal graph in $\mathcal{G}_{n,k}$ belong to $\mathcal{T}'_{n,k}$? We show that the answer to this question, as written in arXiv v2, is no. The failure is at the boundary $n = 2k$ of the range of the construction. For every $k \geq 3$, Construction 9 with $n = 2k$ gives no tree of order $2k$: each tree that it builds has at least $2k + 1$ vertices. The path P_{2k} is in $\mathcal{G}_{2k,k}$, so $\mathcal{G}_{2k,k}$ has spectrally minimal graphs, and none of them is in $\mathcal{T}'_{2k,k}$. The proof is a short degree count. Lean 4 with Mathlib checks the counting step for all $k \geq 3$, with only the standard axioms; the facts about P_{2k} are elementary and are checked by hand and by computer for small k. The result does not touch the range $n \geq 2k + 1$. There, a floating-point search over all trees of order at most 20 finds every spectrally minimal graph in $\mathcal{T}'_{n,k}$ (90 pairs (n, k)), so the question restricted to $n \geq 2k+1$ stays open. We did not read the journal version of the source paper (Linear Algebra Appl. 755 (2027)), so we do not know if its text has the same range. This work has not yet been reviewed by experts. The computations, the proofs and the text were produced by AI models (from Anthropic and OpenAI), working under the author's direction.

1 Introduction

All graphs are finite and simple. The *spectral radius* $\rho(G)$ of a graph G is the largest eigenvalue of its adjacency matrix. The *matching number* of G is the largest number of pairwise disjoint edges of G. Let $\mathcal{G}_{n,k}$ be the family of connected graphs of order n with matching number k. A graph in $\mathcal{G}_{n,k}$ is *spectrally minimal* if its spectral radius is the minimum of ρ over $\mathcal{G}_{n,k}$. A non-empty $\mathcal{G}_{n,k}$ is finite, so it then has at least one spectrally minimal graph.

History. To find the extremal spectral radius in a class of graphs is a problem of Brualdi–Solheid type. For the maximum of ρ on $\mathcal{G}_{n,k}$, Hou and Tan [1] give the history. Liu, Lou and Trevisan [3] started the systematic study of the minimum. They asked for the spectrally minimal graphs of $\mathcal{G}_{n,k}$ (Problem 1 of [1], quoted from [3]). They showed that all spectrally minimal graphs of $\mathcal{G}_{n,k}$ are trees, and they found the minimisers for $k = 2, 3, 4$ by explicit structural formulas.

Hou and Tan [1] proved that every $G \in \mathcal{G}_{n,k}$ satisfies $\rho(G) \geq \sqrt{(n + 2k - 3)/k}$ (their Theorem 3). When $k \mid (n - 3)$, equality holds exactly for the trees of their Construction 2, so this answers Problem 1 in that case. For general n and k they stated a second construction. We quote it from arXiv v2 [1], p. 9:

Construction 9 (General extremal family $\mathcal{T}'_{n,k}$). Let k and n be positive integers with $n \geq 2k$. Define $c := \left\lfloor \frac{n+2k-3}{k} \right\rfloor$. Let T_1 be a tree on vertex set $X = \{x_1, \ldots, x_k\}$ for which there exist nonnegative integers $f_1, \ldots, f_k$ satisfying $2d_{T_1}(x_i) + f_i \in \{c, c+1\}$ for all $i = 1, \ldots, k$. We construct a family of trees $\mathcal{T}'_{n,k}$ as follows:

1. Subdivide every edge of T_1 by inserting a new vertex into each edge.
2. Attach f_i pendant leaves to each x_i, where the f_i's are chosen as above.

The resulting graph is a tree of order n with matching number k (as in the proof of Construction 2).

Directly after it, Hou and Tan wrote:

> Our computational experiments suggest that for general n and k, all spectrally minimal graphs in $\mathcal{G}_{n,k}$ are contained in $\mathcal{T}'_{n,k}$. This leads to the following question: for arbitrary n and k, must every spectrally minimal graph in $\mathcal{G}_{n,k}$ belongs to $\mathcal{T}'_{n,k}$.

Versions of the source. arXiv v1 of [1] (13 July 2026) states Construction 9 for $n \geq 2k - 1$. arXiv v2 (17 July 2026) states it for $n \geq 2k$. The question has the same text in both versions. The case $n = 2k$ is in the range of both. The paper also appeared in Linear Algebra and its Applications 755 (2027) 151–163 [2]. We did not read that text: the publisher blocked our automated access, and its full-text interface gave only metadata. So it is not confirmed that the journal version states Construction 9 for $n \geq 2k$. If it states it only for $n \geq 2k + 1$, our result does not apply to the journal text.

Result. We show that the answer to the question, as written in arXiv v1 and v2, is no (Theorem 1 and Corollary 2). The reason is that Construction 9 gives no tree of order $2k$ when $n = 2k$ and $k \geq 3$. This is a defect at the boundary of the range of the construction. It is not a new spectrally minimal graph: the minimiser in $\mathcal{G}_{2k,k}$ is the path P_{2k} in our computations for $k \leq 10$ (Section 4). The substantial case $n \geq 2k + 1$ stays open. A search over all trees of order at most 20 supports a positive answer there (Section 4).

Prior work on the question. On 27 November 2026 we searched two public collections of AI-produced results, their issue trackers, Zenodo, Semantic Scholar, OpenAlex and the web for a settlement or a discussion of this question. We found none. Semantic Scholar listed no citing work of [1], and OpenAlex listed no citing work of [2].

2 The result

Theorem 1. *Let $k \geq 3$ and $n = 2k$. For every tree T_1 on $X = \{x_1, \ldots, x_k\}$ and all nonnegative integers $f_1, \ldots, f_k$ that satisfy the condition of Construction 9, the tree that steps 1 and 2 build has at least $2k + 1$ vertices. So $\mathcal{T}'_{2k,k}$ contains no graph of order $2k$.*

Corollary 2. *For every $k \geq 3$, $\mathcal{G}_{2k,k}$ has spectrally minimal graphs, and none of them belongs to $\mathcal{T}'_{2k,k}$. So the answer to the question of Hou and Tan is no.*

Proof. The path P_{2k} is connected and has order $2k$. Its edges $v_1v_2, v_3v_4, \ldots, v_{2k-1}v_{2k}$ are k pairwise disjoint edges, and no graph of order $2k$ has more than k pairwise disjoint edges. So $P_{2k} \in \mathcal{G}_{2k,k}$, and $\mathcal{G}_{2k,k}$ is finite and non-empty. Hence it has spectrally minimal graphs. Each of them has order $2k$, so by Theorem 1 none of them is in $\mathcal{T}'_{2k,k}$. □

The last sentence of Construction 9 can be read in two ways. In the first reading, $\mathcal{T}'_{n,k}$ contains only the built trees of order n. Then $\mathcal{T}'_{2k,k}$ is empty. In the second reading, $\mathcal{T}'_{n,k}$ contains all built trees. Then every member has order at least $2k+1$, so no member is a graph in $\mathcal{G}_{2k,k}$. Corollary 2 holds in both readings.

Figure 1 shows the case $k = 3$.

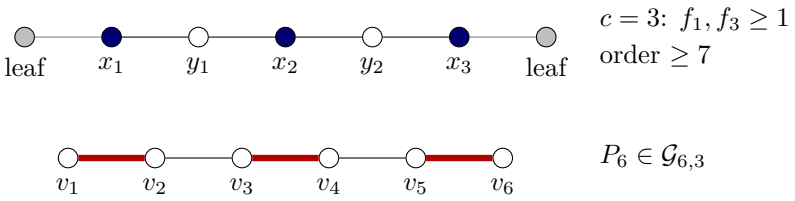


Figure 1: Top: Construction 9 with $k = 3$, $n = 6$ and T_1 the path $x_1x_2x_3$. The ends need $f_1, f_3 \geq 1$, so the built tree has at least 7 vertices. The other tree on three vertices is the same path with other labels. Bottom: the path P_6 with a perfect matching (thick edges).

3 Proof of Theorem 1

Let $k \geq 3$ and $n = 2k$. Write $d_i = d_{T_1}(x_i)$.

1. *The value of c.* Here $c = \lfloor (4k-3)/k \rfloor$. Since $k \geq 3$, we have $3k \leq 4k-3 < 4k$, so $c = 3$.

2. *One vertex.* The condition is $2d_i + f_i \in \{3, 4\}$ with $d_i, f_i \geq 0$. The solutions (d_i, f_i) are $(0, 3)$, $(0, 4)$, $(1, 1)$, $(1, 2)$ and $(2, 0)$. In each solution, $d_i + f_i \geq 2$.

3. *Sum over X.* So $\sum_i d_i + \sum_i f_i \geq 2k$. The tree T_1 has $k-1$ edges, so $\sum_i d_i = 2(k-1)$. Hence $\sum_i f_i \geq 2$.

4. *Order.* Step 1 adds one vertex for each of the $k-1$ edges of T_1. Step 2 adds $\sum_i f_i$ leaves. So the built tree has $k + (k-1) + \sum_i f_i \geq 2k+1$ vertices. □

Remark 3 (small k)**.** The condition $k \geq 3$ is necessary. For $k = 1$ and $n = 2$, $c = 1$, and $f_1 = 1$ gives $K_2 = P_2$. For $k = 2$ and $n = 4$, $c = 2$, $T_1 = K_2$ and $(f_1, f_2) = (1, 0)$ give P_4. In both cases the path is in $\mathcal{T}'_{2k,k}$, and our search confirms that it is the minimiser.

Remark 4 (Construction 2)**.** When $k \mid (n-3)$, the family $\mathcal{T}_{n,k}$ of Construction 2 of [1] is the part of $\mathcal{T}'_{n,k}$ with $2d_i + f_i = c$ for all i. The only pair with $n = 2k$, $k \geq 3$ and $k \mid (n-3)$ is $(n, k) = (6, 3)$, so Theorem 1 also shows that $\mathcal{T}_{6,3}$ has no tree of order 6. This agrees with the remark in [1] after Construction 2, which claims a member of $\mathcal{T}_{n,k}$ only for $k \geq 4$.

Remark 5 (ceiling)**.** The page image of [1] shows floor brackets around $(n + 2k - 3)/k$. With a ceiling instead, $c = 3$ for $k = 3$, and $c = 4$ for $n = 2k$ and $k \geq 4$. In the second case $f_i \geq 4 - 2d_i$, so $\sum_i f_i \geq 4k - 4(k-1) = 4$, and the built tree has at least $2k+3$ vertices. The independent checker tests this reading too (Section 5). The conclusion does not change.

4 Computation

The minimisers of $\mathcal{G}_{n,k}$ are trees [3]. So the program `checks/search.py` takes every tree of order $n \le 20$ (generated by NetworkX [5]), computes its matching number exactly and its spectral radius in floating point (NumPy [5]), and keeps, for each k, the trees whose spectral radius is within 10^{-9} of the minimum. It then builds $\mathcal{T}'_{n,k}$ from every non-isomorphic tree T_1 on k vertices and every admissible f with order n, and it tests each minimiser for membership by a canonical form of the tree.

There are 100 pairs (n, k) with $2 \le n \le 20$. The results are:

- For 8 pairs, a minimiser is outside $\mathcal{T}'_{n,k}$. These are exactly the pairs $(2k, k)$ with $k = 3, \dots, 10$. In each of them $\mathcal{T}'_{n,k}$ has no tree of order n, and the unique minimiser is the path P_{2k} (Table 1).
- For the other 92 pairs, every minimiser is in $\mathcal{T}'_{n,k}$. These are the 90 pairs with $n \ge 2k+1$ and the 2 pairs $(2, 1)$ and $(4, 2)$.
- Four pairs have more than one minimiser within the tolerance: $(13, 4)$ and $(17, 4)$ with 3 trees, $(18, 6)$ with 4 trees, and $(19, 4)$ with 2 trees. All of them are in $\mathcal{T}'_{n,k}$.

k	$n = 2k$	trees built by Construction 9 of order n	$\rho(P_{2k})$
3	6	0	1.801938
4	8	0	1.879385
5	10	0	1.918986
6	12	0	1.941884
7	14	0	1.956295
8	16	0	1.965946
9	18	0	1.972723
10	20	0	1.977662

Table 1: The 8 pairs (n, k) with $n \le 20$ where a minimiser is outside $\mathcal{T}'_{n,k}$. In each, the unique minimiser in the search is P_{2k}, and its computed spectral radius agrees with $2\cos(\pi/(2k+1))$ to within 10^{-9}.

These results are computed in floating point, with a tolerance for ties. They are evidence, not a proof. The proof of Corollary 2 does not use them. They support the following restricted question, which stays open.

Question 6. For $n \ge 2k + 1$, must every spectrally minimal graph in $\mathcal{G}_{n,k}$ belong to $\mathcal{T}'_{n,k}$?

That P_n has the smallest spectral radius among connected graphs of order n is usually attributed to Collatz and Sinogowitz. We did not read their paper, and we do not use this fact.

5 How the result was checked

Lean. The file `lean/Solution.lean` states and proves Theorem 1 in Lean 4 (v4.36.0) with Mathlib (tag v4.36.0) [4]. The statement, which `lean/Challenge.lean` repeats with `sorry` as the proof, is:

```
def Construction9Data (n k : ℕ) (T1 : SimpleGraph (Fin k)) [DecidableRel T1.Adj]
    (f : Fin k → ℕ) : Prop :=
  T1.IsTree ∧ ∀ i, 2 * T1.degree i + f i = (n + 2 * k - 3) / k ∨
    2 * T1.degree i + f i = (n + 2 * k - 3) / k + 1

def construction9Order (k : ℕ) (T1 : SimpleGraph (Fin k)) [DecidableRel T1.Adj]
    (f : Fin k → ℕ) : ℕ :=
  k + #T1.edgeFinset + ∑ i, f i

theorem construction9_no_tree_of_order_two_mul (k : ℕ) (hk : 3 ≤ k)
    (T1 : SimpleGraph (Fin k)) [DecidableRel T1.Adj] (f : Fin k → ℕ)
    (h : Construction9Data (2 * k) k T1 f) : construction9Order k T1 f ≠ 2 * k
```

The match with Construction 9 is as follows. X is `Fin k`, and T_1 is a Mathlib simple graph on it with `IsTree`. The f_i are natural numbers, which include 0. On natural numbers, `/` is the floor of the quotient, and the subtraction $n + 2k - 3$ is exact because $n + 2k \geq 3$. The order of the built tree is the definition `construction9Order`: $|X|$, plus one vertex for each edge of T_1, plus the leaves. Lean does not build the tree itself, and it does not define the matching number, the spectral radius or $\mathcal{G}_{n,k}$. So Lean checks steps 1 to 4 of the proof for all $k \geq 3$. The two facts in the proof of Corollary 2 (that P_{2k} is in $\mathcal{G}_{2k,k}$, and that a finite non-empty family has a minimiser) are checked by hand, and by the Python checkers for small k.

The command `#print axioms` gives

`[propext, Classical.choice, Quot.sound]`.

The proof uses no `native_decide`. The `lake comparator` tool confirms that `Solution.lean` proves exactly the theorem of `Challenge.lean`.

Python. Two checkers were written separately. They are not formally verified.

- `checks/verify.py` uses only the standard library and exact integers. For $k = 3, \ldots, 8$ it takes the degree sequence of every labelled tree on k vertices (from all Prüfer codes) and every admissible f, and it finds no order equal to $2k$. It computes the matching number of P_{2k} by brute force over all edge sets and gets k. As a control, it finds members for $n = 2k+1$.
- `checks/verifier_check.py` builds every graph of Construction 9 explicitly for $k = 3, \ldots, 7$, with both the floor and the ceiling reading of c. The smallest order built is $2k+1$ (floor). For $k = 3, \ldots, 6$ it also takes every tree of order $2k$ with matching number k and searches it for a set X that shows it as a member of $\mathcal{T}'_{2k,k}$. It finds none, and the unique tree of minimum spectral radius is the path. Controls: for $n = 2k+1$ the minimiser is a member, and $P_4 \in \mathcal{T}'_{4,2}$.

6 Limits

1. The result is at the boundary $n = 2k$ of Construction 9. It shows that the stated range of the construction is too large by one value of n for $k \geq 3$. It gives no information about the range $n \geq 2k + 1$, which is the main content of the question.
2. The result answers the question as written in arXiv v1 and v2 of [1]. We did not read the journal version [2]. It is not confirmed that the journal version keeps the range $n \geq 2k$.
3. Lean checks the counting step only. The membership $P_{2k} \in \mathcal{G}_{2k,k}$ and the existence of a minimiser are elementary and are checked by hand.

4. The search of Section 4 uses floating point and a tolerance for ties. Its statements about the pairs with $n \geq 2k+1$ are evidence only.

References

[1] X. Hou, L. Tan, *Tight lower bound for the spectral radius of connected graphs with given matching number*, arXiv:2607.11061, v1 (13 July 2026) and v2 (17 July 2026). `https://arxiv.org/abs/2607.11061`

[2] X. Hou, L. Tan, *Tight lower bound for the spectral radius of connected graphs with given matching number*, Linear Algebra Appl. 755 (2027) 151–163, doi:10.1016/j.laa.2026.10.032. Bibliographic record checked in Crossref; full text not read.

[3] J. Liu, Z. Lou, V. Trevisan, *Minimal spectral radius of graphs with given matching number*, arXiv:2601.18223 (2026), `https://arxiv.org/abs/2601.18223`; Linear Algebra Appl. 746 (2026) 88–110, doi:10.1016/j.laa.2026.05.011.

[4] The Lean 4 theorem prover, `https://github.com/leanprover/lean4`, and the Mathlib library, `https://github.com/leanprover-community/mathlib4`.

[5] NetworkX 3.7, `https://networkx.org`, and NumPy 2.5.5, `https://numpy.org`.